27/02/18 MATH INDUCTION

suppose : $\sum_{i=1}^{100} (2i+1) = 3+5+7+9+11+\dots+201$

This is an arithmetic sequence : difference betw. any 2 consecutive numbers = 2

$$\sum_{i=1}^{100} (2i+1) = \frac{\#\text{ of terms } (1\text{st term} + \text{last term})}{2}$$

$$= \frac{100\,(3+201)}{2} = 10200$$

eg : $\sum_{i=0}^{4} (2i+1) = 1+3+5+7+9 = 25 = \frac{5(1+9)}{2}$

generally, $\sum_{i=3}^{n}$, then number of terms $= (n-3+1) = (n-2)$

or $\sum_{i=m}^{n}$, then number of terms $= (n-m+1)$

question : show $\sum_{i=0}^{n} \cancel{2i} (2i+1) = (n+1)^2$, for every $n \geq 1$.

proof : we prove by Math Induction

$\boxed{1}$ We prove it for $n=1$. (the first possible value of n).

$$\sum_{i=0}^{1} (2i+1) = 1+3 = 4$$

check, when $n=1$, $(n+1)^2 = 2^2 = 4$.

$\boxed{2}$ Assume, it is true for some positive integer $n \geq 1$. This means we assume $\sum_{i=0}^{n} (2i+1) = (n+1)^2$. In other words, we assume what we need to prove is true.

$\boxed{3}$ Prove it for $(n+1)$. I.e $\sum_{i=0}^{n+1} (2i+1) = (n+1+1)^2 = (n+2)^2$.

[key : we must $\boxed{2}$ with some math manipulation to prove $\boxed{3}$]

$$\sum_{i=0}^{n+1} (2i+1) = \sum_{i=0}^{n} (2i+1) + \underbrace{(2n+3)}$$

from : $2(n+1)+1 = 2n+2+1 = 2n+3.$

From $\boxed{2}$, we know $\sum_{i=0}^{n} (2i+1) = (n+1)^2$

So
$$\begin{aligned} \sum_{i=0}^{n+1} &= (n+1)^2 + (2n+3) \\ &= n^2 + 2n + 1 + 2n + 3 \\ &= n^2 + 4n + 4 \\ &= (n+2)^2 \\ &= ((n+1)+1)^2 \end{aligned}$$

If you keep repeating $\boxed{2}$ and $\boxed{3}$ in a loop, this holds true for any value of n.
Hence, this formula is true for any value of $n \geq 1$.

Question: $a_1, a_2, a_3, \dots, a_n$ is a sequence.
let $a_1 = \cancel{4}\ 2$
$a_{n+1} = \sqrt{a_n + 1}$, $\forall\ n \geq 1$
for every.

Prove that $a_n < 3$, $\forall\ n \geq 1$

Proof : $\boxed{1}$ let $n = 1$.
$a_1 = 2$, as given.
$a_{1+1} = a_2 = \sqrt{a_1 + 1} = \sqrt{3}$
and $\sqrt{3} < 3$.

$\boxed{2}$ Assume $a_n < 3\ \forall n \geq 1$.
also, $a_n = \sqrt{a_{n-1} + 1} < 3$

$\boxed{3}$ let $n = k+1$.

$a_{k+1} = \sqrt{a_{k} + 1}$, $a_k < 3$

so, $a_{k+1} < \sqrt{3+1}$
$< \sqrt{4}$
< 2
and $2 < 3$,
hence $a_{k+1} < 3$.

question: Use induction to prove $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$, $\forall\ n \in \mathbb{N}^*$

proof: $\boxed{1}$ prove it for $n = 1$.

$$\sum_{i=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2}$$

and $\dfrac{n}{n+1} = \dfrac{1}{2}$

$\boxed{2}$ Assume it is true for some $n = k \geq 1$

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

$\boxed{3}$ Prove it for $n = k+1$ i.e $\displaystyle\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1+1)} = \frac{k+1}{k+2}$

we agree that $\displaystyle\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+1+1)}$

from step $\boxed{2}$, $\displaystyle\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$

Hence, $\displaystyle\sum_{i=1}^{k+1} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$ ✓

question. $x_1 = 4$, $x_{n+1} = \sqrt{3+4x_n}$. Use math induction to prove that $x_n \leq 5$, $\forall\, n \geq 1$.

proof: [1] Prove for $n=1$.

$x_1 = 4$ and $x_{1+1} = x_2 = \sqrt{3+4x_1} = \sqrt{3+16} = \sqrt{19}$

$\sqrt{19} < 5$

Hence true for $n=1$.

[2] Assume $x_n \leqslant 5$, $\forall\, n \geqslant 1$.

$x_n = \sqrt{3+4x_{n-1}} \leqslant 5$, $\forall\, n \geq 1$.

[3] Prove for $n = k+1$.

$x_{k+1} = \sqrt{3+4x_k} \leqslant 5$

Since we assume $x_k \leqslant 5$,

$x_{k+1} \leqslant \sqrt{3+4(5)}$
$\leqslant \sqrt{23}$

$\sqrt{23} < 5$,

Hence $x_{k+1} \leqslant 5$ ✓

question: (i) $\exists\, x \in \mathbb{N}^*$ and $\exists\, y \in \mathbb{Z}$ s.t $x+y=0$

ans: True, for every positive integer x, there will be a negative integer y, where $|x| = |y|$

Hence $x+y=0$

eg: $x=1$, $y=-1$

$x+y = 1-1 = 0$.

(ii) $\exists\, x \in \mathbb{N}^*$ s.t $x+y=0$, $\forall\, y \in \mathbb{Z}$

✗ ans: False. For a positive integer $y \in \mathbb{Z}$, there can be no positive integer $x \in \mathbb{N}^*$ that can result in $x+y=0$

eg: $y=2$, no x exists where $y+x = 2+x = 0$ with $x \in \mathbb{N}^*$.

(iii) $\exists\, x \in \mathbb{Z}^*$ s.t. $x+y=0$ for every $y \in \mathbb{Z}$.

ans: False. if $y=0$, there can be no non-zero integer $x \in \mathbb{Z}^*$ that can result in $x+y=0$.

eg, $y=0$, $0+x \neq 0$ for any $x \in \mathbb{Z}^*$

1/03/17

question: use math induction to show $4^{4n}-1$ is divisible by 5 $\forall\, n \geq 1$.

i.e. $5 \mid 4^{4n}-1$

i.e $4^{4n} \equiv 1 \pmod 5$

i.e $4^{4n} = 1$ in $\mathbb{Z}_5$

proof: $\boxed{1}$ let $n=1$, prove it.

$4^4 - 1 = 255$. 255 is divisible by 5.

In fact $\frac{255}{5} = 51$, so $255 = 5 \times 51$.

So it is true for $n=1$

$\boxed{2}$ Assume it is true for $n = k \geq 1$.

Hence, we assume that $4^{4n}-1$ is divisible by 5 $\forall\, n = k \geq 1$.

$\boxed{3}$ Prove it for $n = k+1$. We need to show that $4^{4(n+1)} - 1$ is divisible by 5.

$$\begin{aligned} 4^{4(n+1)} - 1 &= 4^{4n+4} - 1 \\ &= 4^{4n} \cdot 4^4 - 1 \\ &= 4^{4n} \cdot 4^4 - 4^4 + 4^4 - 1 \\ &= 4^4(4^{4n}-1) + 4^4 - 1 \end{aligned}$$

from $\boxed{1}$ $= 4^4(4^{4n}-1) + 255$

from $\boxed{2}$: $4^{4n}-1$ is divisible by 5.

so $\underbrace{4^4(4^{4n}-1)}_{\text{divisible}} + \underbrace{255}_{\text{divisible}}$

Hence, $4^{4(n+1)} - 1$ is divisible by 5.

eg: Prove $5 \mid 4^{4n} - 1$ directly.

i.e $4^{4n} \equiv 1 \pmod 5$.

$\phi(5) = 4$, and $a^{\phi(5)} \equiv 1 \pmod 5$

so by Euler Fermat result,

$4^4 \equiv 1 \pmod 5$

let $a = 4^n$

Hence $(4^n)^4 = 1$ in $\mathbb{Z}_5$ also.

or even $(4^4)^n \equiv 1^n \pmod 5$

so $4^{4n} \equiv 1 \pmod 5$.

question: i) use math induction to show that $11^{8n} - 1$ is divisible by 15 $\forall\, n \geq 1$.

ii) use direct proof to show $15 \mid 11^{8n} - 1$

iii) use math induction to show $9^{12n+1} - 9$ is divisible by 13 $\forall\, n \geq 1$. Prove directly also.

* side note: once proved, can also prove $11^{8n+1} \equiv 11 \pmod{15}$.

* if $3^6 \equiv 1 \pmod 7$

then $3^9 \equiv 6 \pmod 7$

etc.

~~Proof: $\boxed{1}$ let $n=1$, prove it~~

logic quantifiers

eg: $x \in \mathbb{Q}$, and hence x might equal to $\sqrt{7}$ — False.

$\in$: belongs to, is in.

$\exists$: there exists (at least one)

$\exists! x$: there exists a unique x. (only one)

$\forall m$: for every m

$\nexists$: does not exist.

$\notin$: does not belong

Question: T or F

i. $\exists x \in \mathbb{N}$ s.t $x^2 - 2 = 0 \Rightarrow$ F (not in $\mathbb{N}$)

ii) $\exists! x \in \mathbb{R}$ s.t $x^2 - 2 = 0 \Rightarrow$ F (not unique)

iii) $\exists x \in \mathbb{R}$ s.t $x^2 - 2 = 0 \Rightarrow$ T (at least one in $\mathbb{R}$).

iv) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t $xy = 1 \Rightarrow$ F (does not apply for 0).

v.) $\forall x \in \mathbb{R}^*$, $\exists y \in \mathbb{R}$ s.t $xy = 1 \Rightarrow$ T

vi) $\forall x \in \mathbb{R}^*$, $\exists y \in \mathbb{Q}$ s.t $xy = 1 \Rightarrow$ F

4/03. use Math Induction to prove that $24 \mid n(n+1)(n+2)(n+3)$, $\forall n \geq 1$.

step 1: Prove for $n=1$.

step 2: Assume for some $n \geq 1$.

step 3: Prove for $n+1$.

show $24 \mid (n+1)(n+2)(n+3)(n+4)$.

i.e show $24 \mid n(n+1)(n+2)(n+3) + 4(n+1)(n+2)(n+3)$.

* We need to show: $6 \mid (n+1)(n+2)(n+3)$ $\forall n \geq 1$.

i.e $6 \mid n(n+1)(n+2) + 3(n+1)(n+2)$.

prove $6 \mid n(n+1)(n+2)$ $\forall n \geq 1$.

$\boxed{1}$ let $n=1$.

$1(2)(3) = 6$, $6 \mid 6$

$\boxed{2}$ Assume for some $n \geq 1$.

i.e $6 \mid n(n+1)(n+2)$ for some $n \geq 1$.

$\boxed{3}$ Prove for $n+1$.

$6 \mid (n+1)(n+2)(n+3)$

$\Rightarrow 6 \mid n(n+1)(n+2) + 3(n+1)(n+2)$

From $\boxed{2}$, we know that $6 \mid n(n+1)(n+2)$

Fact: $(n+1)(n+2)$ is always even $\forall n \geq 1$. i.e $3 \times 2 \mid 3(n+1)(n+2)$

so $6 \mid 3(n+1)(n+2)$. $\Rightarrow 6 \mid (n+1)(n+2)(n+3) \Rightarrow 6 \mid n(n+1)(n+2)$

Now, $24 \mid n(n+1)(n+2)(n+3) + 4(n+1)(n+2)(n+3)$

$6 \times 4 \mid \underbrace{n(n+1)(n+2)(n+3)}_{24 \text{ is a factor}} + \underbrace{4(n+1)(n+2)(n+3)}_{4 \mid 4 \text{ and from above, } 6 \mid (n+1)(n+2)(n+3)}$

$24 \mid (n+1)(n+2)(n+3)(n+4)$. $\Rightarrow 24 \mid n(n+1)(n+2)(n+3)$.

General result: for any $m \in \mathbb{N}$,
$m! \mid n(n+1)(n+2)\cdots(n+m-1)$ $\forall n \geq 1$.

question:

i. $\exists x \in \mathbb{R}^*$ s.t $\forall y \in \mathbb{R}^*$, $xy = 1$.

FALSE. Why? Using the same x for every (all) y. i.e x is independant of y.

ii. $\forall y \in \mathbb{R}^*$, $\exists x \in \mathbb{R}^*$ s.t $xy = 1$.

TRUE. Each and every y will have at least one x.
i.e x depends on value of y.

iii) $\forall y \in \mathbb{R}^*$, $\exists! x \in \mathbb{R}$ s.t $xy = 1$.

TRUE.

* note: the order is important.

iv) $\forall y \in \mathbb{R}^*$, $\exists! x \in \mathbb{Q}^*$ st $xy = 1$.

FALSE. suppose $y = \sqrt{2}$. $x \neq \frac{1}{\sqrt{2}}$ bc $x \in \mathbb{Q}^*$

question: Prove there are infinitely many irrational number. (direct proof).
ans:
- There are infinitely many prime number.
- any $\sqrt{\text{prime}}$ # is irrational. (using rule $\text{prime}^m \mid b^n$ then $\text{prime} \mid b$).

Hence there are infinitely many irrational number.

If ___S1___, then ___S2___ logic:

i.e $S1 \Rightarrow S_2$
↳ implies.

eg. If $x^2 + 1 = -3$, then 3 is irrational number. This is a true statement.
ans: $S1: x^2+1 = -3$ $S2$: 3 is irrational.
S1 is false (positive + positive $\neq$ false).
Since S1 is false, then ~~truth of S2~~ is always true.
↳ the whole statement

IF S1 is true, then the whole statement is true only if S2 is true.
IF S1 is true and S2 is false, the whole statement is false.

logic table:

S1	S2	S1 ⇒ S2
F	x	T
T	T	T
T	F	F

eg: Prove $13 \mid 9^{12n+1} - 9$
ans: $\div 9 \rightarrow 9^{12n} - 1$.
enough to prove $13 \mid 9^{12n} - 1$. direct ~~prove~~ proof.
if proof with induction: $\boxed{1}$ let $n=1$
$\boxed{2}$ Assume.
$\boxed{3}$ Prove for $n+1$.
$\therefore 13 \mid 9^{12n+13} - 9$

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Use math induction to prove $13 \mid 9^{12n+1} - 9$. $\forall n \geq 1$.

$\boxed{1}$ Prove for $n=1$.
$9^{13} - 9$ = some large integer that's divisible by 13,

$\boxed{2}$ Assume $13 \mid 9^{12n+1} - 9$ for some $n \geq 1$.

$\boxed{3}$ Prove for $n+1$.

$9^{12(n+1)+1} - 9$

$= 9^{12n+13} - 9$

$= 9^{12n} 9^{13} - 9^{13} + 9^{13} - 9$

$= 9^{13}(9^{12n} - 1) + 9^{13} - 9$

Multiply by 9; $9^{12}(9^{12n} \cdot 9 - 9) + 9^{13} - 9$

$$9^{12} \underbrace{(9^{12n+1} - 9)}_{\text{divisible by 13, as shown in } \boxed{2}} + \underbrace{9^{13} - 9}_{\text{divisible by 13, as shown in } \boxed{1}}$$

Hence $13 \mid 9^{12n+13} - 9 \Rightarrow 13 \mid 9^{12n+1} - 9 \quad \forall n \geq 1$.

Prove directly.

$\phi(13) = 12$, ~~&~~ $\gcd(9,13) = 1$.
hence $9^{12} \equiv 1 \pmod{13}$
~~Multiply by 9~~; $(9^{12})^n \equiv 1^n \pmod{13}$.
$9^{12n} \cdot 9 \equiv 1 \cdot 9 \pmod{13}$
$\cancel{9^{12n+1}}$
$9^{12n+1} \equiv 9 \pmod{13}$

Hence $13 \mid 9^{12n+1} - 9$.

question: show ~~$17 \mid 18^n + 1$~~ $\forall n \geq 1$. (directly).
$17 \mid 18^n - 1$

i.e $18^n \equiv 1 \pmod{17}$
$18^n = 1$ in $\mathbb{Z}_{17}$
$(1)^n = 1$ in $\mathbb{Z}_{17}$ $\forall n \geq 1$.

Note: 7^9 in $\mathbb{Z}_3$ can be written $(7 \bmod 3)^9$ in $\mathbb{Z}_3$
$(1)^9$ in $\mathbb{Z}_3$.
Cannot do $1^{9 \bmod 3}$.

question: Prove $10 \mid (13^{4n} - 1)$. (directly).
ans: $\phi(10) = 4$. 13, 10 have gcd = 1.
so $13^4 = 1$ in $\mathbb{Z}_{10}$
$\Rightarrow 13^{4n} = 1$ in $\mathbb{Z}_{10}$

Hence $10 \mid (13^{4n} - 1)$.
For free, we get: $10 \mid (3^{4n} - 1)$
Because $13^{4n} \equiv 1 \pmod{10}$
$\Rightarrow (13 \bmod 10)^{4n} = 1$
$\Rightarrow 3^{4n} = 1$ in $\mathbb{Z}_{10}$.

~~question: $\exists!\, x \in \mathbb{N}$ s.t $yx = 4y \quad \forall y \in \mathbb{R}$~~
~~ans: False. If $y = 0$, then x can be anything. x is not unique~~
ignore

[1] $\exists!\ x \in \mathbb{N}$ s.t $yx = 4y \quad \forall y \in \mathbb{R}$.
Translate: There is one $x \in \mathbb{N}$ where $yx = 4y$ for every $y \in \mathbb{R}$.
Ans. True. $x = 4$ will work for every possible value of y. Even when $y = 0$, $x = 4$ will work. No other x will work for every y.

[2] $\forall y \in \mathbb{R}$, $\exists!\ x \in \mathbb{N}$ s.t $yx = 4y$.
Translate: For every $y \in \mathbb{R}$, there is a unique $x \in \mathbb{N}$ st. $yx = 4y$.
Ans: False. What if $y = 0$, x can be any value. not unique.

[3] $\forall y \in \mathbb{R}^*$, $\exists!\ x \in \mathbb{N}$ s.t $yx = 4y$

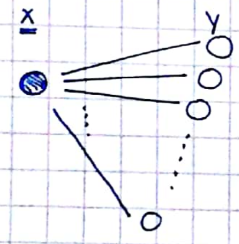
Ans. True.

[4] $\exists x \in \mathbb{N}^*$ s.t. $y^x \geq 0 \quad \forall y \in \mathbb{R}$
Translate. There is at least one x in $\mathbb{N}^*$, s.t whenever y^x the number is always in $\mathbb{N}$, and y can be any number in $\mathbb{R}$.
Ans: True. For x = even number, $y^x \geq 0$

[5] $\exists!\ x \in \mathbb{N}^*$ s.t $y^x \geq 0 \quad \forall y \in \mathbb{R}$
Translate: There is unique natural number x, such that $y^x \geq 0$ for every value of y in $\mathbb{R}$.
Ans: False. x is not unique.

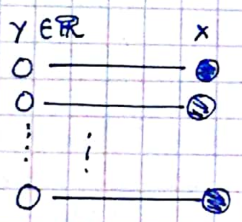
* Mappings:

[1]



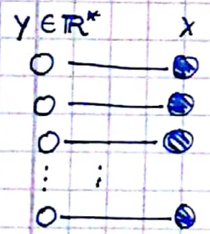
x works for all values of y.
Does it uniquely exist?

[2]

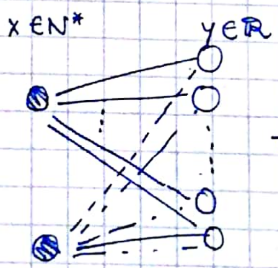


each y has ONE value of x such that $yx = 4y$
Does it uniquely exist?

[3]

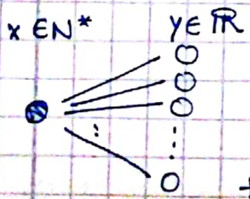


[4]



there is ~~only~~ atleast one x
for all y.
~~Two~~ questions: does it exist
~~is it unique~~.

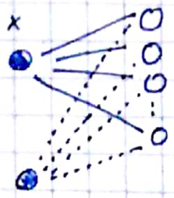
[5]



there is one x
for all y
two questions: exist?
unique?

$\boxed{5}$ $\exists x \in \mathbb{Q}^*$ s.t $xy = 1$ $\forall y \in \mathbb{Q}^*$

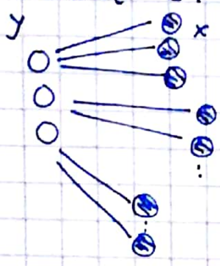
Mapping :



False.

$\boxed{7}$ $\forall y \in \mathbb{Q}^*$ $\exists x \in \mathbb{Q}^*$ s.t $xy = 1$.

Mapping :



True.

* Note : eg $8^{17} \mid a^3$
then

$8 \nmid a$ bc. 8 not prime.

but $8^{17} \mid a^3$

$\Rightarrow (2^3)^{17} \mid a^3$

$\Rightarrow 2 \mid a$ bc. 2 is prime.

general rule (don't use ~~en~~ in exam) :
if m is a number, such that m is a product of distinct primes, and
$m^k \mid n^p$ then $m \mid n$.